GENERALIZED GEOMETRIES

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PHYSICS CONCEPTS

- 3-form flux
- gauged sigma-models with WZ term
- D-branes
- skew torsion

GENERALIZED GEOMETRIES

BASIC SCENARIO

- manifold M^n
- replace T by $T \oplus T^*$
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- manifold M^n
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$$(X + \xi, X + \xi) = i_X \xi$$

- skew adjoint transformations:

$$\mathrm{End}\, T \oplus \Lambda^2 T^* \oplus \Lambda^2 T$$

- .... *in particular* $B \in \Lambda^2 T^*$

B-FIELD TRANSFORMATIONS

- exponentiate B:

  $X + \xi \mapsto X + \xi + i_X B$

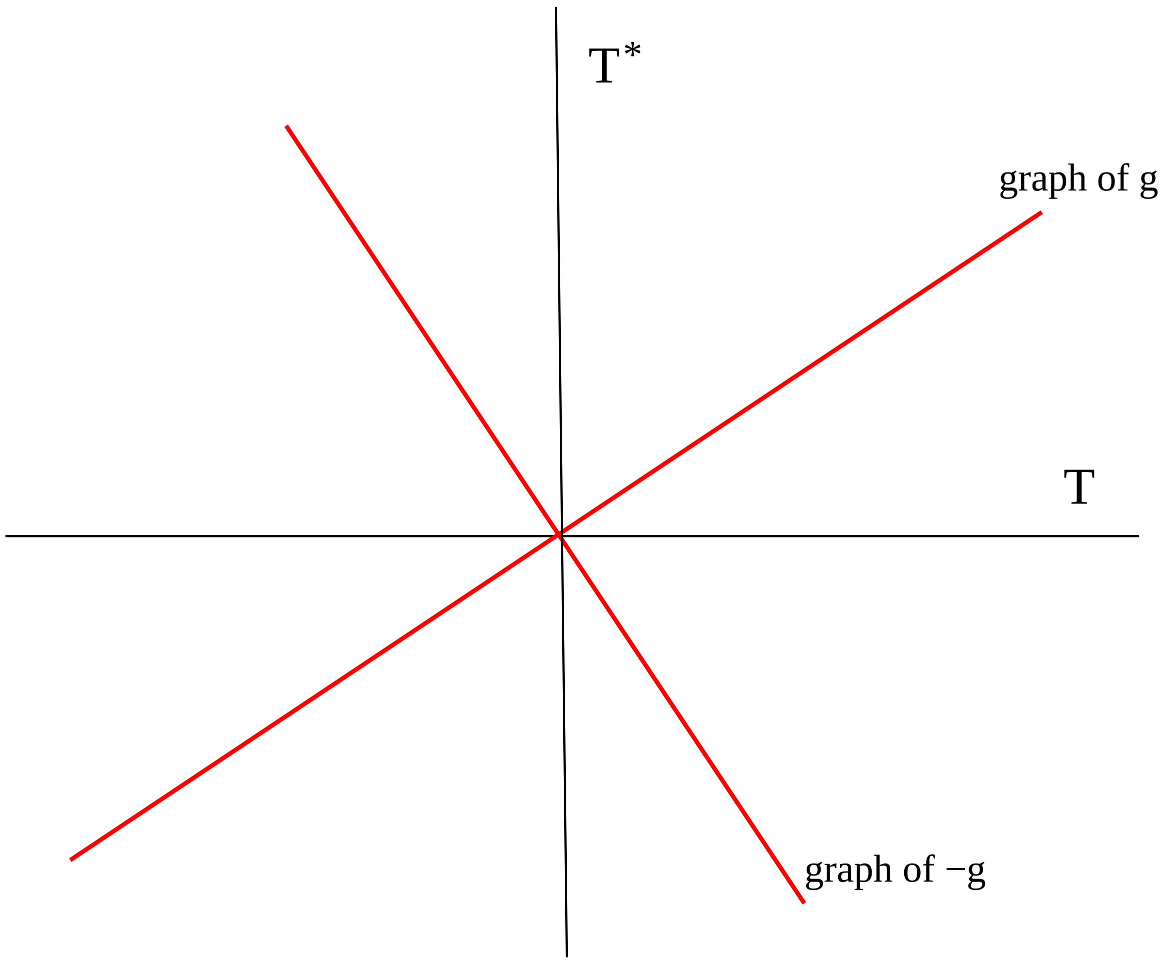
- this is an orthogonal transformation of $T \oplus T^*$

GENERALIZED METRICS

RIEMANNIAN METRIC

- Riemannian metric g_{ij}
- $X \mapsto g(X,-) : g : T \to T^*$
- *graph* of $g = V \subset T \oplus T^*$
- $X + gX \in V$, inner product

$$(X + gX, X + gX) = g(X, X)$$


T*
graph of g
T
graph of −g

GENERALIZED RIEMANNIAN METRIC

- $V \subset T \oplus T^*$ positive definite rank n subbundle

- $=$ graph of $g + B : T \to T^*$

- $g + B \in T^* \otimes T^*$: g symmetric, B skew

THE COURANT BRACKET

Bracket on sections of $T \oplus T^*$

- $[X+\xi, Y+\eta] = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi)$
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- $\pi(u)(v,w) = ([u,v] + d(u,v), w) + (v, [u,w] + d(u,w))$

  where $\pi(X+\xi) = X$

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- $[u, fv] = f[u,v] + (\pi(u)f)v - (u,v)df$

- $\pi(u)(v,w) = ([u,v] + d(u,v), w) + (v, [u,w] + d(u,w))$

  where $\pi(X+\xi) = X$

- preserved by $X+\xi \mapsto X+\xi+i_XB$ if B is *closed*

AFFINE CONNECTIONS

- A generalized metric defines *two* subbundles V and $V^{\perp}$ of $T \oplus T^*$

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- $[X^-, Y^+] - [X, Y]^-$ is a one-form

- $[X^-, Y^+] - [X, Y]^- = 2g\nabla_X Y$

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- ∇ Riemannian connection with skew torsion $-H/2$ $(H = dB)$

- $[X^+, Y^-] - [X,Y]^+$ has skew torsion $H/2$

EXAMPLE: the Levi-Civita connection

$$\left[\frac{\partial}{\partial x_i} - g_{ik}dx_k, \frac{\partial}{\partial x_j} + g_{jk}dx_k\right] - \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right]^{-} =$$

$$= \left(\frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ik}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k}\right) dx_k = 2 g_{\ell k} \Gamma^{\ell}_{ij} dx_k$$

EXAMPLE: connection with torsion dB

$$\left[\frac{\partial}{\partial x_i} - g_{ik}dx_k + B_{ik}dx_k, \frac{\partial}{\partial x_j} + g_{jk}dx_k + B_{jk}dx_k\right] =$$

$$= \left(\frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ik}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k}\right) dx_k + \left(\frac{\partial B_{jk}}{\partial x_i} - \frac{\partial B_{ik}}{\partial x_j}\right) dx_k$$

TWISTING WITH A GERBE

GERBES

- $g_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \to S^1$

- $(g_{\alpha\beta\gamma} = g^{-1}_{\beta\alpha\gamma} = \ldots)$

- $\delta g = g_{\beta\gamma\delta} g^{-1}_{\alpha\gamma\delta} g_{\alpha\beta\delta} g^{-1}_{\alpha\beta\gamma} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$

TRIVIALIZATIONS

- $g_{\alpha\beta\gamma} = h_{\alpha\beta}h_{\beta\gamma}h_{\gamma\alpha}$

- $\tilde{h}_{\alpha\beta}h_{\alpha\beta}^{-1}$ are the transition functions for a unitary line bundle

- "the ratio of two trivializations is a line bundle"

CONNECTIONS ON GERBES I

- Connective structure:

  $$A_{\alpha\beta} + A_{\beta\gamma} + A_{\gamma\alpha} = g_{\alpha\beta\gamma}^{-1} dg_{\alpha\beta\gamma}$$

- Flat trivialization: $A_{\alpha\beta} = h_{\alpha\beta}^{-1} dh_{\alpha\beta}$

- $\Rightarrow$ line bundle on loop space

CONNECTIONS ON GERBES II

- Curving:

  $B_\beta - B_\alpha = dA_{\alpha\beta}$

- $\Rightarrow dB_\beta = dB_\alpha = H|_{U_\alpha}$ global three-form H

- J.-L. Brylinski, *Characteristic classes and geometric quantization*, Progr. in Mathematics **107**, Birkhäuser, Boston (1993)

TWISTING $T \oplus T^*$

$$dA_{\alpha\beta} + dA_{\beta\gamma} + dA_{\gamma\alpha} = d[g^{-1}_{\alpha\beta\gamma} dg_{\alpha\beta\gamma}] = 0$$

- patch $T \oplus T^*$ over U_α with $T \oplus T^*$ over U_β with

  $X + \xi \mapsto X + \xi + i_X dA_{\alpha\beta}$

- defines a vector bundle E

  $0 \to T^* \to E \to T \to 0$

- with ... an inner product and a Courant bracket.

Definition: A **generalized metric** is a subbundle $V \subset E$ such that $\mathrm{rk}\, V = \dim M$ and the inner product is positive definite on V.

- $V \cap T^* = 0 \Rightarrow$ splitting of the sequence

  $0 \to T^* \to E \to T \to 0$

SPLITTINGS IN LOCAL TERMS

- splitting: $C_\alpha \in C^\infty(U_\alpha, T^* \otimes T^*)$: $C_\beta - C_\alpha = dA_{\alpha\beta}$
- $Sym(C_\alpha) = Sym(C_\beta) =$ metric
- $Alt(C_\alpha) = B_\alpha =$ curving of the gerbe
- $H = dB_\alpha$ closed 3-form

SUBMANIFOLDS

- $Y \subset M$ submanifold

- $TY \oplus N^*Y \subset T \oplus T^*$ generalized tangent bundle

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- twisted version?

- assume the gerbe has a flat trivialization $h_{\alpha\beta}$ on Y...

- ... then $A_{\alpha\beta} = h_{\alpha\beta}^{-1} dh_{\alpha\beta}$ on Y...

- and so $dA_{\alpha\beta} = 0$ on Y, so if $X \in TY$, $\xi \in N^*Y$,

  $X + \xi + i_X dA_{\alpha\beta} \in TY \oplus N^*Y$

- choice of flat trivialization $\sim$ flat line bundle

TWISTED COHOMOLOGY

SPINORS

- Take $S = \Lambda^{\bullet} T^*$

- $S = S^{ev} \oplus S^{od}$

- Define Clifford multiplication by

$$\begin{aligned}(X+\xi)\cdot\varphi &= i_X\varphi + \xi\wedge\varphi \\ (X+\xi)^2\cdot\varphi &= i_X\xi\varphi = (X+\xi, X+\xi)\varphi\end{aligned}$$

- $\exp B(\varphi) = (1 + B + \frac{1}{2}B\wedge B + \ldots)\wedge\varphi$

- spinor bundle S for $0 \to T^* \to E \to T \to 0$

- patch $\Lambda^\bullet T^*$ on U_α to $\Lambda^\bullet T^*$ on U_β with

  $\varphi \mapsto (\exp dA_{\alpha\beta}) \wedge \varphi$

- $d : C^\infty(S) \to C^\infty(S)$

- twisted cohomology

- $Y \subset M$ with a flat trivialization of the gerbe

- restrict φ_α to Y

- $\varphi_\beta = (\exp dA_{\alpha\beta}) \wedge \varphi_\alpha = \varphi_\alpha$ on Y

- integrate $\Rightarrow$ Y defines a twisted homology class

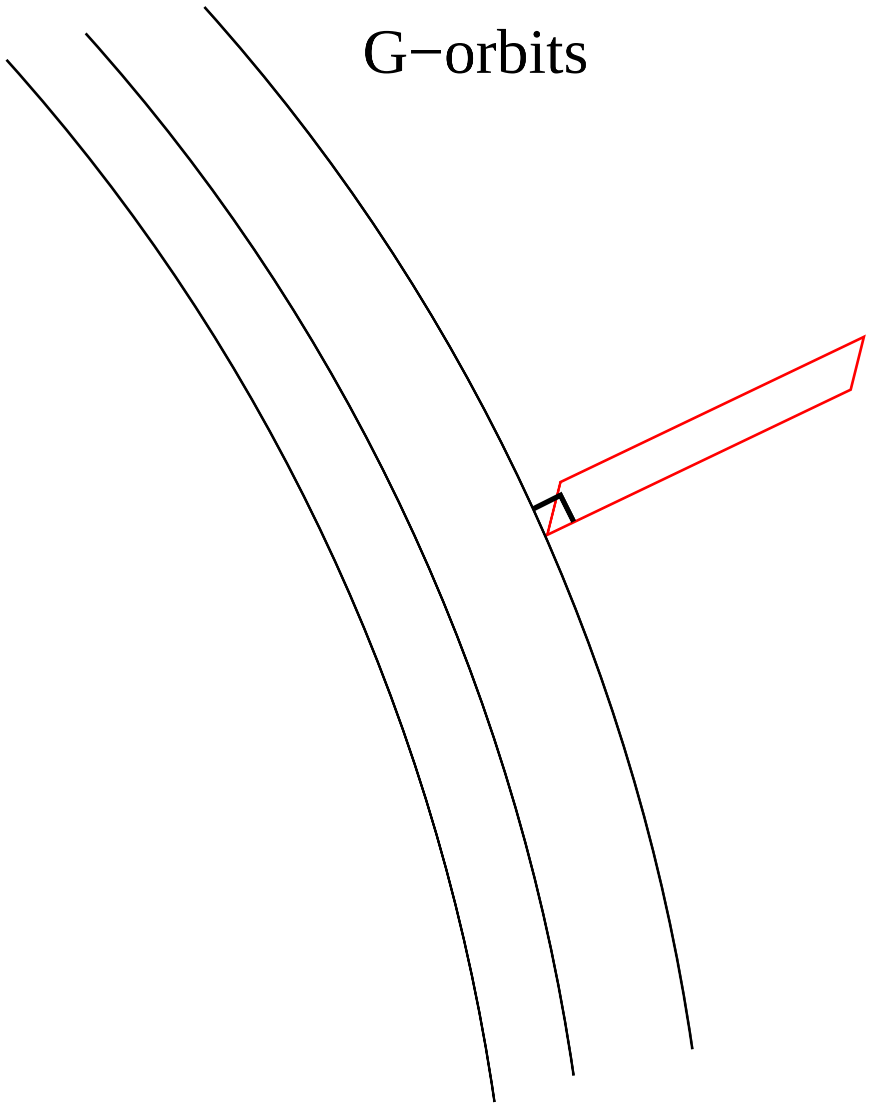
QUOTIENTS

- Riemannian manifold M

- proper free action of a group G of isometries

- M/G is a manifold

- M/G has an induced Riemannian metric


G−orbits

- M with a *generalized* metric

- proper free action of a group G of isometries

- M/G is a manifold

- Does M/G have a generalized metric?

- "gauged sigma model with Wess-Zumino term"

ACTIONS

THE COURANT ACTION

- Lie algebra homomorphism of $\mathfrak{g}$ to vector fields

- lift X to a section $e_X = X + \xi_\alpha$ of $0 \to T^* \to E \to T \to 0$

- $\xi_\beta - \xi_\alpha = i_X dA_{\alpha\beta}$

PROPERTIES NEEDED

- $e_{[X,Y]} = [e_X, e_Y]$ (Courant bracket)

- $(e_X, e_X) = 0$

- $\Rightarrow$ rank g ($=$ dim G) isotropic subbundle $K \subset E$

ACTION OF G on E

Define a "Lie derivative"

$$L_X(Y+\eta_\alpha) = \mathcal{L}_X(Y+\eta_\alpha) - i_Y d\xi_\alpha$$

$$\mathcal{L}_X(Y+\eta_\beta) = \mathcal{L}_X(Y+\eta_\alpha + i_Y dA_{\alpha\beta})$$

$$= [X,Y] + \mathcal{L}_X\eta_\alpha + i_{[X,Y]}dA_{\alpha\beta} + i_Y(d\xi_\beta - d\xi_\alpha)$$

- "Dorfmann bracket"

- $L_{[X,Y]} = [L_X, L_Y]$

DEFINING THE QUOTIENT

- $K \subset K^{\perp}$
- $\dim K^{\perp}/K = (2n - g - g) = 2\dim M/G$
- $\bar{E} = (K^{\perp}/K)/G$ is a bundle on M/G and

$$0 \to T^*(M/G) \to \bar{E} \to T(M/G) \to 0$$

- .... with non-degenerate inner product and Courant bracket.

THE QUOTIENT GENERALIZED METRIC

- generalized metric $V \subset E$, positive definite

- $V \cap K^{\perp} \to K^{\perp}/K$

- K isotropic $\Rightarrow V \cap K = 0 \Rightarrow$ injective

- $\dim(V \cap K^{\perp}) = 2n - \dim(V^{\perp} + K) = n - g$

- $\Rightarrow$ generalized metric on M/G.

THE MOMENT FORM

- $B_\beta - B_\alpha = dA_{\alpha\beta}$
- $\xi_\beta - \xi_\alpha = i_X dA_{\alpha\beta}$
- $c = i_X B_\alpha - \xi_\alpha = i_X B_\beta - \xi_\beta$ well-defined one-form
- $c \in \Omega^1(M) \otimes \mathfrak{g}^*$ in general

- $\mathcal{L}_X B_\alpha = d\xi_\alpha$ (invariance of generalized metric)

- $dc = -d\xi_\alpha + di_X B_\alpha = -i_X dB_\alpha = -i_X H$

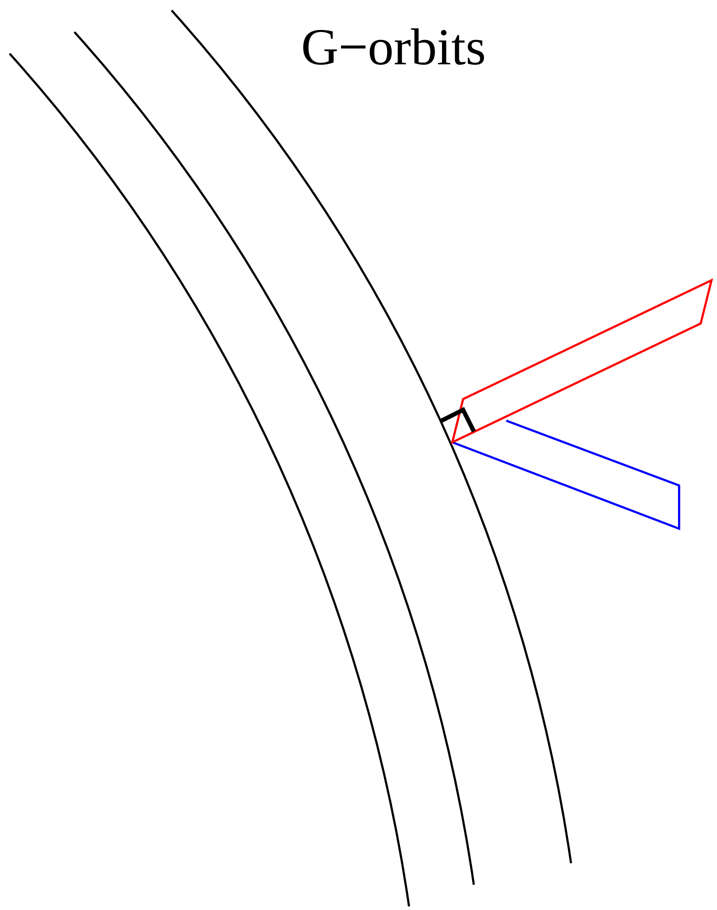
THE QUOTIENT METRIC

- $V \cap K^{\perp}$

- $0 = (Y + gY + i_Y B_\alpha, X + \xi_\alpha) = g(Y,X) + B_\alpha(Y,X) + \xi_\alpha(Y)$

  $\Rightarrow g(Y,X) - c(Y) = 0$

- new horizontality condition


G−orbits

THE THREE-FORM H

- suppose $dB = 0$ on M, then ...
- ... M/G can have a non-trivial three-form:
- horizontals define a connection on $M \to M/G$...
- .... with curvature $F \in \Omega^2(M, \mathfrak{g})$
- $c \in \Omega^1(M, \mathfrak{g}^*)$ and $H = -\langle c, F\rangle \in \Omega^3(M)$

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