Stochastic Loewner Evolution: another way of thinking about Conformal Field Theory

John Cardy

University of Oxford

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Centre for Mathematical Physics, Hamburg

Outline

- recall some facts about 2d CFT
- what kind of field theories are being described, and their lattice versions
- the basics of SLE
- conformal restriction measures
- identification of the stress tensor and derivation of the Ward identities of $c = 0$ CFT
- extension to $c > 0$: conformal loop ensemble (CLE)

Conformal Field Theory

- massless, renormalised 2d euclidean QFT
- local operators which transform simply under conformal transformations $z \to f(z)$:

$$\phi(z,\bar{z}) \to f'(z)^{\Delta_\phi}\, \overline{f'(z)}^{\overline{\Delta}_\phi}\, \phi\big(f(z), \overline{f(z)}\big)$$

- stress tensor $T(z)$ generates infinitesimal conformal transformations $z \to z + \alpha(z)$ via insertion of

$$\int_C \frac{dz}{2\pi i} \alpha(z) T(z) + \text{c.c.}$$

  into correlation functions
- equivalent to OPEs

$$T(z) \cdot \phi(z_1, \bar{z}_1) = \frac{\Delta_\phi}{(z-z_1)^2} \phi(z_1, \bar{z}_1) + \frac{1}{z-z_1} \partial_{z_1} \phi(z_1, \bar{z}_1) + \cdots$$

$$T(z) \cdot T(z_1) = \frac{c/2}{(z-z_1)^4} + \frac{2}{(z-z_1)^2} T(z_1) + \frac{1}{z-z_1} \partial_{z_1} T(z_1) + \cdots$$

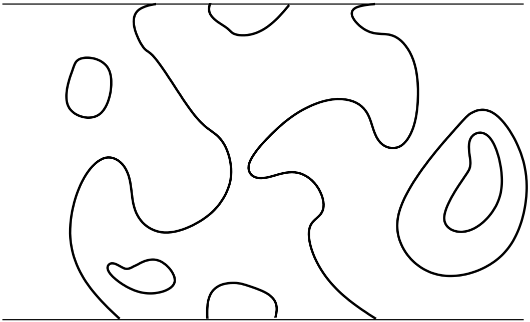
Example: $O(n)$ scalar field theory

- n-component field Φ_j, bare action

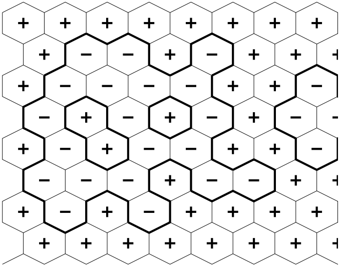
$$S = \int \Big[\sum_{j=1}^{n} \big((\partial \Phi_j)^2 + m_0^2 \Phi_j^2 \big) + \lambda_0 \Big(\sum_{j=1}^{n} \Phi_j^2 \Big)^2 \Big] d^2 r$$

- critical point at $m_R^2 = 0$ for $n \leq 2$
- RG fixed point at $\lambda_0 \to \infty$
- world-lines of particles do not cross

Space-imaginary time picture

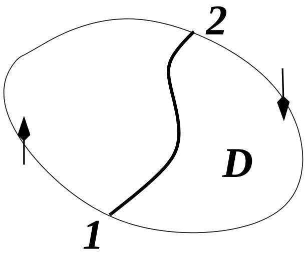


Lattice version

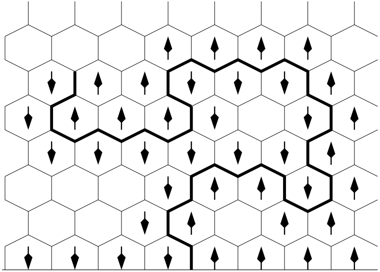


- gas of non-intersecting loops and open curves weighted by their total length, factor n for each closed loop, eg
  - $n = 1$: Ising model
  - $n = 2$: dual to Kosterlitz-Thouless transition
  - $n = 0$: self-avoiding walks (“quenched approximation”)

- in the continuum limit (at critical point) these loops become fractal curves - what is the measure on these?
  - or, what is the measure on just one of them?
  - specify conditions on the boundary of a simple connected domain $\mathcal{D}$ such that there is always a single open curve from r_1 to r_2:



- such curves can be ‘grown’ on the lattice by a discrete exploration process:



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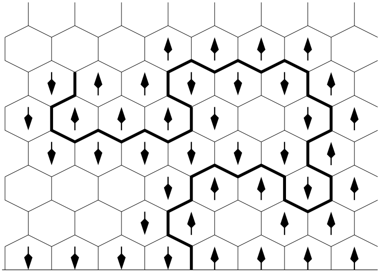
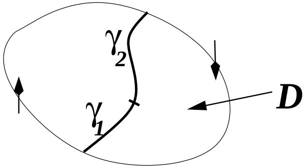


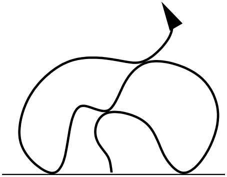
- SLE describes the continuous version of this

The postulates of SLE



- Denote the curve by γ, and divide it into two disjoint parts.
- conditional measure on γ_2 given γ_1 is the same as the unconditional measure on γ_2 in $\mathcal{D} \setminus \gamma_1$
- moreover this is conformally related to the measure on γ in $\mathcal{D}$

- choose $\mathcal{D}$ = upper half plane $\mathbf{H}$
- let K_t be the curve + all the regions enclosed by it at time t



- let $g_t(z)$ be the conformal mapping which sends $\mathbf{H} \setminus K_t$ to $\mathbf{H}$, normalised so that

$$g_t(z) \sim z + 0 + \frac{2t}{z} + \cdots \qquad (\text{as } z \to \infty)$$

- g_t sends the growing tip into a_t on the real axis
- the evolution of g_t satisfies the Loewner equation

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - a_t}$$

- if curve is continuous so is a_t
- so instead of thinking about a measure on curves we can think about a measure on continuous functions a_t

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Theorem. [Schramm] *If above postulates hold then a_t is proportional to a standard Brownian motion.*
That is

$$a_t = \sqrt{\kappa}\, B_t$$

so that $\langle a_t \rangle = 0$, $\langle \left(a_{t_1} - a_{t_2}\right)^2 \rangle = \kappa |t_1 - t_2|$.

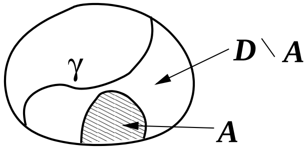
- one-parameter family of conformally invariant measures on curves labelled by κ
- many boundary and bulk scaling dimensions can be derived rigorously from the postulates of SLE `[Lawler-Schramm-Werner]`
- stochastic process $\Rightarrow$ Fokker-Planck equations (2nd order PDEs) $\rightleftharpoons$ BPZ differential equations of CFT following from condition that boundary field Φ_j satisfies $L_{-2}\Phi_j \propto L_{-1}^2\Phi_j$ `[Bauer-Bernard]`

$$n = -2\cos(4\pi/\kappa) \qquad (2 \leq \kappa \leq 8)$$

$$\text{central charge} \quad c = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa}$$

- how we identify the stress tensor T for these random curves?
- can we derive the conformal Ward identities?

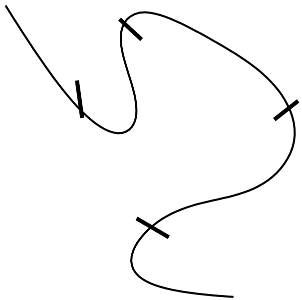
- start with the simplest case $n = 0$ ("quenched approximation")
- satisfies *conformal restriction*



- measure on γ restricted not to lie in A is the same as the measure we get by conformally mapping $\mathcal{D} \to \mathcal{D} \setminus A$
- expect this to be true for $n = 0$ but not in general, because 'vacuum processes' are sensitive to A.
- Theorems (1)`[L-S-W]` SLE satisfies this only for $\kappa = \frac{8}{3}$; (2)`[Werner]` there is a unique measure on single self-avoiding *loops* which satisfies restriction

What is the stress tensor?

- its trace measures response to a dilatation (so vanishes at an RG fixed point)
- its trace-free part measures response to a local shear, i.e. the local anisotropy of the medium
- in 2d it has two independent components $(T,\overline{T})$ which have 'spin' ± 2: under $z\to ze^{i\theta}$, $T\to e^{-2i\theta}T$, $\overline{T}\to e^{2i\theta}\overline{T}$
- leads to the following guess:



- slits of lengths $\{\epsilon_j\}$, at angles $\{\theta_j\}$, centred on points $\{z_j\}$
- let

$$P\big(\{\epsilon_j\},\{\theta_j\},\{z_j\}\big) = \Pr\big(\gamma \text{ intersects every slit}\big)$$

and let

$$Q\big(\{z_j\}\big) = \lim_{\epsilon_j\to 0}\prod_j \epsilon_j^{-2}\prod_j\int\frac{d\theta_j}{2\pi}e^{-2i\theta_j}P(\ldots)$$

Theorem. [Doyon-Riva-JC]: the limit exists and satisfies the conformal Ward identities as

$$\langle T(z_1)T(z_2)\ldots\rangle = \sum_{j\neq 1}\left(\frac{h_j}{(z_1-z_j)^2}+\frac{1}{z_1-z_j}\partial_{z_j}\right)\langle T(z_2)\ldots\rangle$$

(with $c=0$).

- *Proof*: based on conformal restriction applied to the probabilities that γ avoids subsets of the slits.

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- *Proof*: based on conformal restriction applied to the probabilities that γ avoids subsets of the slits.
- by conditioning γ also to pass around given points $\{\zeta_j\}$ and taking limits as they coincide, can generate a whole set of local fields which form a closed operator algebra
- $\Rightarrow$ complete and rigorous construction of the whole CFT

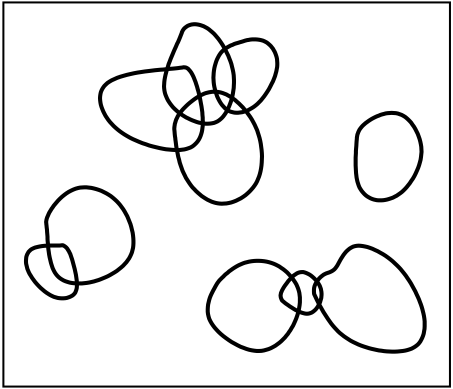
How do we make CFTs with $c > 0$?

Conformal Loop Ensemble
[Werner-Lawler-Sheffield]:

- start with the (unique) measure on single self-avoiding loops
- partition function

$$Z \propto \int^{L} \frac{dR}{R} \sim \ln L$$

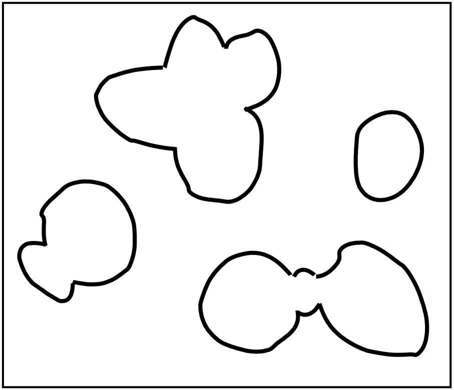
- let them rain down independently and uniformly for a 'time' τ



- for small enough τ these form disjoint clusters

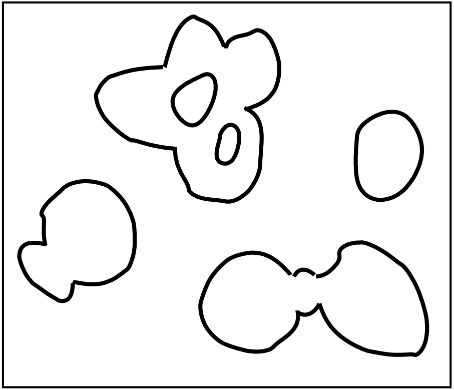
$$Z \sim e^{\text{const.}\tau \ln L}$$

- look only at the outermost boundaries:



- these should be the same as the outermost set of loops in the $O(n)$ model for $n > 0$

- to get the full nested set, fill them iteratively



- none of this changes Z, so

$$\text{central charge} \quad c = \text{const.}\ \tau$$

- if τ too large, get one big cluster ($\Rightarrow c > 1$)
- if we take $T \propto$ spin-2 component of probability that *any* loop in this construction crosses a small slit, we get the full conformal Ward identity with the central term $\propto c$.

Summary

- SLE and its extensions give a (rigorous) geometrical picture of the continuum limit of systems which should also be described by CFT
- conformal invariance is manifest
- in the simplest case of conformal restriction we can identify the stress tensor and derive the Ward identities of a $c = 0$ CFT
- we can define a complete set of local correlation functions and show they satisfy expected OPEs
- by using the CLE we should be able to extend this to theories with $c > 0$